

The Type B Temperley Lieb Algebra

Cailan Li

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1. Preliminaries

Recall the Dynkin diagram for type B_n .



Definition 1.1. $TL(B_n)$ is defined as

$$TL(B_n) = \mathcal{H}(B_n) \Big/ \left\langle v^{-\ell(w_0(s,t))} b_{w_0(s,t)} \right\rangle_{\forall s,t \text{ distinct}}$$

where $w_0(s,t)$ is the longest element in the parabolic subgroup generated by s, t .

Lemma 1.2 (KL presentation). $\mathcal{H}(B_n)$ is generated as an algebra by $\{b_1, \dots, b_n\}$ with relations

$$b_s^2 = (v + v^{-1})b_s \tag{1}$$

$$b_s b_t = b_t b_s \quad \text{if } |s - t| > 1 \tag{2}$$

$$b_s b_t b_s - b_s = b_t b_s b_t - b_t \quad \text{if } |s - t| = 1 \text{ and } \{s, t\} \neq \{1, 2\} \tag{3}$$

$$b_1 b_2 b_1 b_2 - 2b_1 b_2 = b_2 b_1 b_2 b_1 - 2b_2 b_1 \tag{4}$$

Remark. Setting $e_i = b_i$ in $TL(B_n)$ we see that $\langle e_2, \dots, e_n \rangle$ form a copy of TL_n as $b_s b_t b_s - b_s = b_{sts}$ when $|s - t| = 1$ in this region so $e_i = U_i$ [draw] the standard TL generator. Thus we see that in $TL(B_n)$ the obstruction to a diagrammatic presentation is the relation Eq. (4) as $b_s b_t b_s b_t - 2b_s b_t = b_{stst}$ which in $TL(B_n)$ reads

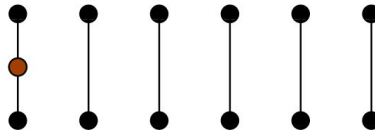
$$e_1 e_2 e_1 e_2 = 2e_1 e_2 \tag{5}$$

as setting $e_1 = U_1$ diagrammatically does not produce the relation above.

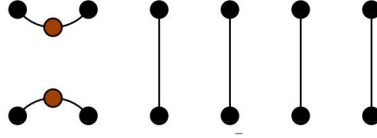
2. Blob Algebra

Decorated tangles $\mathbb{DT}_n$ are tangles (that do not cross) that can carry dots on them with n nodes to n nodes. Edges that link a node on the top with one on the bottom are called propagating.

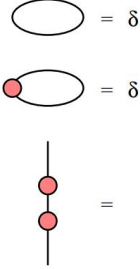
Definition 2.1. The decorated tangle e has all propagating straight lines with a dot on the first line. An example for $\mathbb{DT}_6$ is shown defined below.



Definition 2.2. The decorated tangle $\overline{e_1} = e e_1 e$ is obtained by adding a dot to the two non-propagating edges in e_1 . An example for $\mathbb{DT}_6$ is shown defined below.



Definition 2.3. The blob algebra $b_n(\delta, \delta')$ is the algebra on n strands generated by $\{e, e_1, \dots, e_{n-1}\}$ modulo the relations.



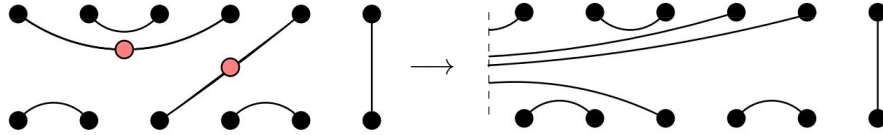
Proposition 2.4 (Martin, Saleur). A basis for $b_n(\delta, \delta')$ is given by blob diagrams which is an element of $\mathbb{DT}_n$ such that

- Only lines exposed to the west face of the diagram may be decorated.
- There is at most one dot on each edge and no loops.

Proof. We just note that the first condition is because the relations of TL_n preserve the dot on stand 1 being exposed to the west face of the diagram and the second is a result of the relations above. ■

Definition 2.5. The asymmetric representation of a blob diagram D = Break each decorated edge of D at the decoration, and connect all the loose endpoints to the west wall in such a way that they do not intersect each other.

Example 1. On the left is a blob diagram D and on the right is the associated asymmetric representation



Definition 2.6. The symmetric representation of $D \in b_n$ is an element of TL_{2n} gotten by reflecting the asymmetric representation for D in the west wall.

Proposition 1 (Dieck)

Symmetric diagrams in TL_{2n} form a subalgebra of dimension $\binom{2n}{n}$. Furthermore there is an isomorphism $\psi : b_n(\delta, 1) \rightarrow \text{Sym}(TL_{2n})(\delta^2)$ sending $\psi(e_i) = e_{n+i}e_{n-i}$, $\psi(e) = e_n/\delta$.

Proof. We just show the bijection between blob diagrams in b_n and symmetric diagrams in TL_{2n} . The symmetric representation gives the map in one direction, for the other truncate a symmetric diagram by one half. We now have a total of $2n$ nodes and each edge occupies two nodes. It follows that the number of nodes that are connected to the west wall is even and thus the number of nodes on the top connected to the west wall has the same parity p as on the bottom.

If $p = 0$, then going from right to left turn each pair of adjacent lines to the west wall from the bottom into caps with a dot on them and similarly with the top and into cups with dots on them. If $p = 1$, turn the rightmost line from the top and the rightmost line from the bottom into a single line with a dot on it. Now repeat with the rest of the diagram as in the $p = 0$ case. One can check that this process is inverse to taking the symmetric representation of D (because only lines exposed to the west face of the diagram D may be decorated). ■

Remark. The root system C_n can be gotten from A_{n-1} by folding [draw auto] and [Proposition 1](#) is sort of a manifestation of this for TL algebras¹. Also, the type B coxeter group can also be realized as symmetric permutation diagrams in S_{2n} .

Given $x \in b_n(\delta, \delta')$, a reduced word is a monomial in the generators of minimal length.

Definition 2.7. A reduced word e_w satisfies the B -condition if

- (1) Neither e_1 nor e occurs in e_w , OR
- (2) Both e_1 and e occur in e_w but e_1 can only occur in between the leftmost and rightmost e .

Lemma 2.8. e commutes with all the generators except e_1 .

***Proposition 2.9.** Let e_w be a reduced word satisfying the B -condition. Then e_w is equal to a word in the generators $\{\bar{e}_1, e_2, \dots, e_{n-1}\}$

Proof. If e_w satisfies (1) we are done. Otherwise it satisfies (2). If there is only e_1 in between the e 's we are done as [Lemma 2.8](#) implies we can slide the e 's to get $ee_1e = \bar{e}_1$. Thus assume there's multiple e_1 's. We claim there has to be at least one e in the area S , where $e_1 \overset{S}{\cdots} e_1$. There must be an element in the S area that doesn't commute with e_1 or else e_w not reduced. Since e_1 commutes with everything except e_2 and e , if there isn't an e in the S area, then there is a single e_2 . But $e_1e_2e_1 = e_1$ so then e_w is not reduced. Thus there has to be at least one e in the S area.

Replace each e with e^2 . Now for each e_1 we can commute the two surrounding e 's to e_1 to form $ee_1e = \bar{e}$ as desired. ■

3. Diagrammatics for $TL(B_n)$

Definition 3.1. Let $DTL(B_n)(\delta, \delta')$ be the subalgebra of $b_{n+1}(\delta, \delta')$ generated by $\{\bar{e}_1, e_2, \dots, e_n\}$.

Theorem 2

$\Phi : TL(B_n) \xrightarrow{\sim} DTL(B_n)([2], [2]/2)$ where $\Phi(e_1) = 2\bar{e}_1$ and $\Phi(e_i) = e_i$ for $i > 1$.

Proof. Φ is surjective and one checks that Φ is a homomorphism with the key relation being $\Phi(e_1e_2e_1e_2) = 2\Phi(e_1e_2)$. Being a subalgebra of $b_{n+1}(\delta, \delta')$,

Lemma 3.2. As a set, $DTL(B_n)$ equals all diagrams of $\mathbb{DT}_n$ where either

- (1) Node 1 in the top is joined to node 1 in the bottom by an undecorated edge, and thus there are no decorated edges.

¹As a Coxeter group, B_n is the same as C_n and thus their Hecke algebras are the same.

- (1') Node 1 in the top is joined to node 1 in the bottom by an decorated edge, with no other decorated edges, but at least one non-propagating edge.
- (2) The edges emerging from node 1 in the north face and node 1 in the south face are distinct and both decorated.

Proof. $\subseteq$ If $D \in DTL(B_n)$ has dots, note that dots only come from $\bar{e}_1$. All other generators except e_2 do not interact with the 2nd nodes on top and bottom, the dot can only possibly move under pre or post composition by e_2 . For $e_2\bar{e}_1e_2$ we get (1') and for $\bar{e}_1e_2\bar{e}_1, e_2\bar{e}_1, \bar{e}_1e_2$ we get (2). (if we had e and e_1 we could get dots on other strands [draw])

$\supseteq$ Given a diagram D in the form (1), it's in the span of $\{e_2, \dots, e_n\}$ so we are done. Otherwise if D is in the form (1') or (2) we have that $D = eDe$. e doesn't commute past e_1 and any reduced form for D satisfies the B -condition and applying [Proposition 2.9](#) we are done. ■

Lemma 3.3. *The number of diagrams of type (1) is C_n , of type (1') or (2) is $\binom{2n}{n} - 1$ where C_n is the n -th Catalan number and thus $DTL(B_n)$ has dimension $(n+2)C_n - 1$*

Proof. (1) Delete the first edge and what's left is a diagram in TL_n which has dimension C_n .

(1') or (2) Because the edge from node 1 on both the top and bottom of D has to have a dot, it follows that in the symmetric representation of D in TL_{2n+2} , $n+1$ is always connected to $n+2$ on both the top and bottom. If we delete these four nodes, we end up with a bijection from diagrams in (1') or (2) to symmetric diagrams in TL_{2n} except the identity. By [Proposition 1](#) this has dimension $\binom{2n}{n} - 1$ as desired. ■

Finally, it is known that $\dim_{\mathbb{k}} TL(B_n) = \text{Type } B_n \text{ Catalan number} = \text{number of fully commutative elements in } B_n = (n+2)C_n - 1$ and thus we are done. ■

Remark. If one uses the Hecke algebra with unequal parameters, the above theorem still holds with δ, δ' arbitrary.